Week 3 Worksheet

Instructions. Discuss with your group mates and do the following problems. You are not expected to finish all the problems. This worksheet is for both Tuesday and Thursday. **Please make sure to bring this worksheet on Thursday! :)**

1 Slope of the tangent line

Question 0. $f(x) = \sqrt{x}$. What is the slope of the tangent line to $y = f(x)$ at $x = 4$? Please select all the correct expressions.

(a) $\lim_{x \to 0} \frac{\sqrt{x}-2}{x-4}$ (b) $\lim_{x \to 4} \frac{\sqrt{x}-4}{x-4}$ (c) $\lim_{\Delta x \to 0} \frac{\sqrt{x}-2}{x-4}$ (d) $\lim_{\Delta x \to 0} \frac{\sqrt{x}-2}{\Delta x}$

(e) $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ (f) $\lim_{\Delta x \to 0} \frac{\sqrt{4+\Delta x}-2}{\Delta x}$ (g) $\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$ (h) $\lim_{\Delta x \to 0} \frac{\sqrt{4+\Delta x}-2}{\Delta x-4}$

2 $\varepsilon - \delta$ definition of limit

The $\epsilon - \delta$ definition of limits is confusing! In this part of the worksheet, we will try to break it down and understand it better. First the definition:

" We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, we can find some $\delta > 0$ such that if $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$. "

Question 1. What does the expression $0 < |x - a| < \delta$ really mean? If you are not sure, interpret it for a concrete value of a and $\delta = .5$. Every time you see the expression $|x - a|$ you can now replace it with this phrase!

The distance between x and a is smaller than δ, and $x \neq a$.

There are two other ways to interpret this definition:

Interpretation 1: A question *"If we want $f(x)$ to differ from L by no more than ϵ, then how close should x be to a?* The answer to this question is δ!

Interpretation 2: A game You are playing the Calculus Games against me. If you lose, I will point and laugh at you. I will give you a function $f(x)$, numbers a, L and a number ϵ. You have to give me back a number δ so that if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

In both cases, your goal given an ϵ is to give back a δ.

Here is a general approach to find δ given ε as a number:

1. Write down $|f(x) - L| < \varepsilon$
2. Solve for x and obtain $x_1 < x < x_2$
3. Determine δ by choosing the smallest distance between x and a, that is, the smaller value of $|x_1 - a|$ and $|x_2 - a|$.

Question 2: We say that $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$, we can find some $\delta > 0$ such that if $0 < |x-a| < \delta$ we have $|f(x) - L| < \epsilon$. Now $f(x) = 2x+1$, $a = 5$, $\varepsilon = 1$. Please find:

(a) the value of L

(b) a possible value of δ

(c) the largest value of δ

(a) $L = \lim_{x\to 5} 2x+1 = 11$

(b) $\frac{1}{10^{10}}$ anything that is extremely small.

(c) $|2x+1-11| < 1$

$-1 < 2x-10 < 1$

$9 < 2x < 11$

$4.5 < x < 5.5$ $\quad\delta = \min(|4.5-5|, |5.5-5|) = 0.5$

Question 3: $\lim_{x\to 2} 5 - 2x = 1$.

(a) What is $f(x)$, a and L in this set-up?

(b) When $\varepsilon = 1$, find the largest δ.

(c) When $\varepsilon = 2$, find the largest δ.

(d) Given a general ε, can you take a guess of δ (as some expression of ε)? Think about why it works.

(a) $f(x) = 5-2x$ $\quad a=2$ $\quad L=1$

(b) $|5-2x-1| < 1$

$|4-2x| < 1$

$-1 < 4-2x < 1$

$-5 < -2x < -3$

$\frac{5}{2} > x > \frac{3}{2}$

$\left|\frac{5}{2}-2\right| = 0.5$

$\left|\frac{3}{2}-2\right| = 0.5$

$\delta = \min(0.5, 0.5) = 0.5$

(c) $|5-2x-1| < 2$

$|4-2x| < 2$

$-2 < 4-2x < 2$

$-6 < -2x < -2$

$3 > x > 1$

$|3-2| = 1$

$|1-2| = 1$

$\delta = 1$

(d) Guess $\delta = \frac{\varepsilon}{2}$.

Need to Check:

If $0 < |x-a| < \delta$, indeed we have $|f(x) - L| < \varepsilon$.

That is:

If $0 < |x-2| < \delta = \frac{\varepsilon}{2}$, we have $|4-2x| < \varepsilon$.

Is above statement true? $\quad |-a| = |a|$

Notice because $|4-2x| = |-(2x-4)| = |2x-4|$

$|4-2x| = |2x-4| = 2|x-2| < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$.

So the expression we guess works!

Question 4. $\lim_{x\to1} \sqrt{3x+1}+1 = 3$. When $\varepsilon = 1$, what is the largest possible value of δ.

$$\left|\sqrt{3x+1}+1-3\right| < 1$$

$$\left|\sqrt{3x+1}-2\right| < 1$$

$$-1 < \sqrt{3x+1}-2 < 1$$

$$1 < \sqrt{3x+1} < 3$$

$$1 < 3x+1 < 9$$

$$0 < 3x < 8$$

$$0 < x < \frac{8}{3}$$

$$|0-1| = 1, \quad \left|\frac{8}{3}-1\right| = \frac{5}{3}$$

$$\delta = \min\left(1, \frac{5}{3}\right) = 1$$

To think about:

Say, for some problem, after solving for x we obtain $\sqrt{3} < x < \sqrt{5}$

And $a = 2$, then what is δ?

Well... It's hard to tell $|\sqrt{3}-2|$ and $|\sqrt{5}-2|$ which is smaller without calculators.

So you can just write:

$$\delta = \min(|\sqrt{3}-2|, |\sqrt{5}-2|).$$

3 Properties of Limits

Question 5. True or False.

F (a) If $\lim_{x\to0} f(x) = 0$ and $\lim_{x\to0} g(x) = 0$, then $\lim_{x\to0} \frac{f(x)}{g(x)}$ does not exist.

F (b) If $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ both do not exist, then $\lim_{x\to0} f(x)+g(x)$ also does not exist.

F (c) If $\lim_{x\to0} f(x)$ exists and $\lim_{x\to0} g(x)$ does not exist, then $\lim_{x\to0} f(x)+g(x)$ could still exist.

T (d) If $\lim_{x\to0} f(x)$ exists and $\lim_{x\to0} g(x)$ does not exist, then $\lim_{x\to0} f(x)\,g(x)$ could still exist.

T (e) If $\lim_{x\to0} f(x)$ exists and $\lim_{x\to0} g(x)$ does not exist, then $\lim_{x\to0} \frac{f(x)}{g(x)}$ could still exist.

F (f) If $\lim_{x\to0} f(x)$ does not exists and $\lim_{x\to0} g(x)$ exists, then $\lim_{x\to0} \frac{f(x)}{g(x)}$ could still exist.

(a) $\lim_{x\to0} \frac{x}{x} = 1$

(b) $\lim_{x\to0} \frac{1}{x}$ and $\lim_{x\to0} -\frac{1}{x}$ DNE but $\lim_{x\to0} \frac{1}{x} - \frac{1}{x} = 0$

(d) $\lim_{x\to0} x = 0 \qquad \lim_{x\to0} \frac{1}{x}$ DNE

But $\lim_{x\to0} x \cdot \frac{1}{x} = 1$ exists!

(e) $\lim_{x\to0} x = 0 \qquad \lim_{x\to0} \frac{1}{x}$ DNE

$\lim_{x\to0} \frac{x}{\frac{1}{x}} = \lim_{x\to0} x^2 = 0$ exists!

[P.S. these T/F Problems came from our textbook. :)]